

# A Characterization of Uniqueness of Limit Models in Categorical Abstract Elementary Classes

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## Abstract

In this paper we examine the task set forth by Shelah and Villaveces in [14] of proving the uniqueness of limit models of cardinality  $\mu$  in  $\lambda$ -categorical abstract elementary classes with no maximal models, where  $\lambda$  is some cardinal larger than  $\mu$ . In [16] and [17] we identified several gaps in the approach outlined in [14], and we added the assumption that the union of an increasing chain of limit models is a limit model.

Here we replace this assumption with the seemingly weaker statement that the union of an increasing and continuous chain of limit models is an amalgamation base. Moreover, we prove that this assumption is not only sufficient but is necessary to settle the uniqueness of limit models problem attempted in [14] for  $\lambda = \mu^{+n}$  when  $0 < n < \omega$ .

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## 1. Introduction

A conjecture guiding Shelah's program of the classification of abstract elementary classes (AECs) since its introduction in the 1970s is a generalization of Loś Conjecture [10]:

**Conjecture 1** (Shelah's Categoricity Conjecture [13]). *If an AEC  $\mathcal{K}$  is categorical in some sufficiently large cardinal  $\lambda$ , then  $\mathcal{K}$  is categorical in all sufficiently large  $\mu$ .*

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There are over a thousand pages of work towards proving this conjecture. For a more complete description of progress on this conjecture see [5, 16, 22, 2]. The approximations to proving the categoricity conjecture involve assuming either additional model-theoretic properties of the class  $\mathcal{K}$  (e.g. homogeneity, finitary properties,  $L_{\kappa,\omega}$ -axiomatizability, amalgamation property) and/or assuming additional set-theoretic assumptions (e.g. existence of large cardinals, GCH). For instance, in [11] Makkai and Shelah assume that the AEC  $\mathcal{K}$  is axiomatizable by a  $L_{\kappa,\omega}$  sentence where  $\kappa$  is a strongly compact cardinal and prove a categoricity transfer theorem if the categoricity cardinal  $\lambda^+$  is  $> \beth_{(2^\kappa)^+}$ . In [12] Shelah proves a downward categoricity transfer theorem from a sufficiently large successor cardinal assuming the amalgamation property. Grossberg and VanDieren transfer categoricity upward from a successor cardinal in a tame AEC that satisfies the amalgamation property [7]. Complementing these results, under the assumption of tameness, Vasey [22] improves the bound from [12] on the categoricity cardinal. It remains open whether or not one can transfer categoricity without tameness and/or without the assumption that the categoricity cardinal is a successor in AECs that satisfy the amalgamation property.

The amalgamation property seems to be key in proving categoricity transfer results. In fact in 1986, Grossberg conjectured that the amalgamation property follows from categoricity [5]:

**Conjecture 2.** *If  $\mathcal{K}$  is an abstract elementary class categorical in a sufficiently large cardinality, then  $\mathcal{K}$  satisfies the amalgamation property.*

Some progress has been made on this conjecture. Kolman and Shelah prove that amalgamation follows from categoricity in  $L_{\kappa,\omega}$ -axiomatizable AECs where  $\kappa$  is a measurable cardinal [9].

The property of the uniqueness of limit models is a stepping stone to both derive the amalgamation property [9] and to prove categoricity transfer results (e.g. [7, 12]) in categorical abstract elementary classes (AECs). Shelah and Villaveces endeavor to prove the uniqueness of limit models in categorical AECs with no maximal models [14]. They use set-theoretic assumptions to derive the density of amalgamation bases from categoricity, and then they attempt to prove the uniqueness of limit models.

**Claim 1** (The main claim, Theorem 3.37, of [14]). *Let  $\mu$  and  $\lambda$  be cardinals so that  $\text{LS}(\mathcal{K}) \leq \mu < \lambda$ . Suppose that GCH and  $\Phi_{\mu^+}(S_{\text{cf}(\mu)}^{\mu^+})$  hold.*

*If  $\mathcal{K}$  is  $\lambda$ -categorical and has no maximal models, then if  $M$  and  $M'$  are limit models of cardinality  $\mu$  over  $M_0$ , then  $M$  and  $M'$  are isomorphic over  $M_0$ .*

While Shelah and Villaveces' work inspired several papers examining the uniqueness of limit models in non-categorical classes as a step to develop a classification theory for non-elementary classes [8, 24, 23, 3, 1, 18, 20, 21], the main result stated in [14] remains open.

In this paper we continue the work begun by Shelah and Villaveces. We identify an assumption that is not only sufficient, but is necessary, to prove Claim 1 for  $\lambda = \mu^{+n}$  where  $0 < n < \omega$ . We also explain the role of the assumptions of *GCH* and  $\Phi_{\mu^+}(S_{\text{cf}(\mu)}^{\mu^+})$ .

Progress on the uniqueness of limit models has not been smooth [16, 17]. In the PhD thesis [15], we identify errors in Shelah and Villaveces' proof of Claim 1 in [14] of the uniqueness of limit models of categorical abstract elementary classes with no maximal models. In [16], we continue to fill these gaps. Two of the problems addressed in [16] are the extension property for towers and the characterization of limit models using relatively full towers.

In [16], we introduce a few alternative assumptions to negotiate the gaps in the proof in [14] of the extension property for towers. We assume that the union of an increasing and continuous chain of limit models is a limit model [16, Hypothesis 2] or that the class of amalgamation bases of cardinality  $\mu$  is closed under unions of sequences of length  $< \mu^+$  [16, Hypothesis 3]. Another assumption that we show to be sufficient to prove the extension property for towers is:

**Assumption 1** (Hypothesis 1 from [16]).<sup>1</sup> *Let  $\mu \geq \text{LS}(\mathcal{K})$  and fix  $\mathfrak{C}$  a  $(\mu, \mu^+)$ -limit model. Every continuous tower inside  $\mathfrak{C}$  has an amalgamable extension inside  $\mathfrak{C}$  (see Section 2 for the definitions).*

Here we replace Assumption 1 with a variation of Hypothesis 2 and 3 of [16] which is strong enough to imply Assumption 1, but is on the surface weaker than Hypothesis 2 and 3 of [16].

**Assumption 2.**<sup>2</sup> *The union of an increasing and continuous chain of limit models  $\langle M_i \in \mathcal{K}_\mu \mid i < \alpha < \mu^+ \rangle$  is an amalgamation base.*

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<sup>1</sup>This is not a global assumption in our paper.

<sup>2</sup>This is not a global assumption in the paper. It will be explicitly stated when used.

Years after the publication of [16], the paper [17] acknowledges another problem with the proofs in [14] and [16], specifically the proof that reduced towers are continuous. We resolve this problem here.

In this paper we show that if the main theorem of [14] is to hold when  $\mathcal{K}$  is categorical in a successor cardinal, then Assumption 2 is necessary. We state this theorem using the set-theoretic assumptions of [14], plus additional instances of the weak diamond that are needed to work with limit models of different cardinalities. In Remark 1 we indicate how these set-theoretic assumptions can be replaced with model-theoretic assumptions and/or eliminated.

**Theorem 1.** *Let  $\mu = \kappa^+$  be cardinal so that  $\text{LS}(\mathcal{K}) \leq \kappa$ . Fix  $n$  a natural number larger than zero and set  $\lambda = \mu^{+n}$ . Suppose that GCH holds and assume  $\Phi_{\chi^+}(S_{\text{cf}(\chi)}^{\chi^+})$  for every  $\chi$  satisfying  $\kappa \leq \chi < \lambda$ . If  $\mathcal{K}$  is  $\lambda$ -categorical and has no maximal models, then the following are equivalent:*

- ① *The union of an increasing and continuous chain of limit models  $\langle M_i \in \mathcal{K}_\mu \mid i < \alpha < \mu^+ \rangle$  is an amalgamation base (Assumption 2).*
- ② *If  $M$  and  $M'$  are limit models of cardinality  $\mu$  over  $M_0$ , then  $M$  and  $M'$  are isomorphic over  $M_0$ .*
- ③ *If  $M$  and  $M'$  are limit models of cardinality  $\mu$ , then  $M$  and  $M'$  are isomorphic.*
- ④ *The union of an increasing chain of saturated models dense with  $\kappa$ -amalgamation bases  $\langle M_i \in \mathcal{K}_\mu \mid i < \alpha < \mu^+ \rangle$  is saturated.*

**Remark 1.** The assumptions of GCH and  $\Phi_{\chi^+}(S_{\text{cf}(\chi)}^{\chi^+})$  in Theorem 1 are used in three places:

- GCH is in the proof of superstability [14, Theorem 2.2.1] which we describe how to eliminate in Remark 3.
- Another use of GCH is to get limit models of each cardinality. But if we do not have limit models of cardinality  $\mu^+$ , then the statement of the theorem is vacuously true. The subtle point where we still use GCH is that if the theorem isn't vacuously true because we have limit models of cardinality  $\mu^+$ , then we will still need to use limit models of cardinality  $\mu$  to prove the theorem. And, without assuming the full

amalgamation property, it is unknown if  $\mu$  stability and the existence of limit models of cardinality  $\mu^+$  are enough to imply  $\mu$  stability or the density of limit models of cardinality  $\mu$ .

- Finally, the diamond-like property,  $\Phi_{\chi^+}(S_{\text{cf}(\chi)}^{\chi^+})$ , is used to show that limit models of cardinality  $\chi$  are amalgamation bases. While the conclusion of the theorem only involves models of cardinality  $\mu = \kappa^+$ , the proofs employ limit models of cardinality  $\kappa$  and models of cardinality larger than  $\mu$  but smaller than  $\lambda$ .

Theorem 1 improves the main result of [17] which is the implication ①  $\Rightarrow$  ② of Theorem 1 for  $\lambda = \mu^+$ , but it also gives us more insight into Claim 1:

**Corollary 1.** *Assumption 2 is necessary and sufficient to prove Claim 1 when  $\lambda = \mu^{+n}$ .*

We begin with some preliminary definitions and results. Section 3 outlines the structure of the intended proof of Claim 1 in [14]. Section 4 explains how to negotiate saturated models when the amalgamation property is not assumed. Then, in Section 5 we work on the implication ①  $\Rightarrow$  ② of Theorem 1. We confirm that the error in the proof that reduced towers are continuous mentioned in [17] can be addressed by proving  $\mu$ -symmetry. We verify that the proofs in the series of papers [18] and [19] can be adapted to this setting in which the full amalgamation property is not assumed. From this we get not only the equivalence of  $\mu$ -symmetry and the statement that reduced towers are continuous, but also the fact that  $\mu^+$ -categoricity implies  $\mu$ -symmetry. We then adopt [20] to this setting to transfer the  $\mu^{+(n-1)}$ -symmetry down to  $\mu$ . Finally, Section 6 contains the remainder of the proof of Theorem 1.

We tackle the adaptation of the proofs from [20] and [21] in an upcoming paper which will be used to improve Theorem 1 by requiring only that  $\mu < \lambda$ .

## 2. Background

For the history of the literature surrounding the uniqueness of limit models and the preliminary definitions and notation (e.g. abstract elementary classes, Galois-types, stability,  $\Phi_{\mu^+}(S_{\text{cf}(\mu)}^{\mu^+})$ , etc.), we refer the reader to [16], [8], and [2]. Here we will review a few of the concepts that we use explicitly in the proof of Theorem 1.

Although we will not have the full amalgamation property at our disposal in this paper, we do have enough amalgamation to carry out several arguments. Here we recall the level of amalgamation that we are guaranteed in the context of Theorem 1.

**Definition 1.** An *amalgamation base* is a model  $M \in \mathcal{K}_\mu$  for which any two models of cardinality  $\mu$  extending  $M$  in  $\mathcal{K}$  can be amalgamated. That is for every  $M_1, M_2 \in \mathcal{K}_\mu$  with  $M \prec_{\mathcal{K}} M_1, M_2$ , there is  $M^* \in \mathcal{K}_\mu$  and  $\mathcal{K}$ -embeddings  $f_1$  and  $f_2$  so that the following diagram commutes:

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & M^* \\ \text{id} \uparrow & & \uparrow f_2 \\ M & \xrightarrow{\text{id}} & M_2 \end{array}$$

The set-theoretic assumption  $\Phi_{\mu^+}(S_{\text{cf}(\mu)}^{\mu^+})$  along with categoricity above  $\mu$  imply the density of amalgamation bases of cardinality  $\mu$ :

**Fact 1** (Theorem 1.2.5 of [14] or see Lemma 1.2.23 of [16]). *Suppose that  $\Phi_{\mu^+}(S_{\text{cf}(\mu)}^{\mu^+})$  holds. Assume that  $\mathcal{K}$  is categorical in  $\lambda$  and  $\mu < \lambda$ .*

*Then for every  $M \in \mathcal{K}_\lambda$  and  $N \prec_{\mathcal{K}} M$  of cardinality  $\mu$ , there exists an amalgamation base  $N' \in \mathcal{K}_\mu$  with  $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M$ .*

**Definition 2.** For  $\mu \geq \text{LS}(\mathcal{K})$  and  $\theta$  a limit ordinal  $< \mu^+$ , we say that  $M \in \mathcal{K}_\mu$  is a  $(\mu, \theta)$ -*limit model* if there exists an increasing and continuous sequence of amalgamation bases  $\langle M_i \in \mathcal{K}_\mu \mid i < \theta \rangle$  so that  $M = \bigcup_{i < \theta} M_i$  and  $M_{i+1}$  is universal over  $M_i$ . In this case we say that  $M$  is a  $(\mu, \theta)$ -limit model over  $M_0$ . We also say  $M$  is a *limit model* if there is a limit ordinal  $\theta < \mu^+$  for which  $M$  is a  $(\mu, \theta)$ -limit model.

In the context of Theorem 1, limit models are amalgamation bases:

**Fact 2** (Fact 1.3.10 of [14] or Theorem 1.3.13 of [16]). *Suppose that  $\mathcal{K}$  has no maximal models and is categorical in  $\lambda$  and that  $\mu$  is a cardinal with  $\lambda > \mu \geq \text{LS}(\mathcal{K})$ . Assume that GCH holds. Then any limit model of cardinality  $\mu$  is an amalgamation base. Additionally, for every amalgamation base  $M \in \mathcal{K}_\mu$  and for every limit ordinal  $\theta < \mu^+$ , there exists a  $(\mu, \theta)$ -limit model  $M'$  over  $M$ .*

By  $\mu^+$ -many repeated applications of Fact 2, for any amalgamation base  $M \in \mathcal{K}_\mu$  we can find a  $(\mu, \mu^+)$ -limit model over  $M$ . This model is saturated and will serve as a replacement for a monster model. We will use  $\mathfrak{C}$  to denote such a model in the following sections.

**Remark 2.** Note that if  $M$  and  $M'$  are  $(\mu, \theta)$ - and  $(\mu, \theta')$ -limit models, respectively, over  $M_0$  and  $\text{cf}(\theta) = \text{cf}(\theta')$ , then by a back-and-forth construction,  $M$  and  $M'$  are isomorphic over  $M_0$ . Therefore Claim 1 is only interesting when  $\text{cf}(\theta) \neq \text{cf}(\theta')$ .

Next we recall the definition of the dependence relation that we will be using throughout this paper:  $\mu$ -splitting.

**Definition 3.** For  $M \in \mathcal{K}_\mu$  an amalgamation base and  $p \in \text{gaS}(M)$ , we say that  $p$   $\mu$ -splits over  $N$  iff  $N \prec_\mathcal{K} M$  and there exist amalgamation bases  $N_1, N_2 \in \mathcal{K}_\mu$  and a  $\prec_\mathcal{K}$ -mapping  $h : N_1 \cong N_2$  such that

1.  $N \prec_\mathcal{K} N_1, N_2 \prec_\mathcal{K} M$ ,
2.  $h(p \upharpoonright N_1) \neq p \upharpoonright N_2$  and
3.  $h \upharpoonright N = \text{id}_N$ .

While  $\mu$ -splitting is not as versatile as forking, it does have the extension and uniqueness properties:

**Fact 3** (Theorem I.4.10 of [16]). *Suppose that  $M \in \mathcal{K}_\mu$  is an amalgamation base and universal over  $N$  and  $M'$  is an extension of  $M$  of cardinality  $\mu$  inside  $\mathfrak{C}$ . If  $\text{ga-tp}(a/M)$  does not  $\mu$ -split over  $N$  and there exists  $g \in \text{Aut}_M(\mathfrak{C})$  so that  $\text{ga-tp}(g(a)/M')$  does not  $\mu$ -split over  $N$ .*

**Fact 4** (Theorem I.4.12 of [16]). *Suppose that  $N, M, M' \in \mathcal{K}_\mu$  are amalgamation bases with  $M'$  universal over  $M$  and  $M$  universal over  $N$ . If  $p \in \text{gaS}(M)$  does not  $\mu$ -split over  $N$ , then there exists a unique  $p' \in \text{gaS}(M')$  such that  $p'$  extends  $p$  and  $p'$  does not  $\mu$ -split over  $N$ .*

The uniqueness of limit models is related to the statement that the union of saturated models is saturated, which in first order model theory is equivalent to superstability. Therefore we will be considering  $\mu$ -superstable abstract elementary classes: We will use the following definition of  $\mu$ -superstability:

**Definition 4.**  $\mathcal{K}$  is  $\mu$ -superstable if  $\mathcal{K}$  is Galois-stable in  $\mu$  and  $\mu$ -splitting satisfies the property: for all infinite  $\alpha < \mu^+$ , for every sequence  $\langle M_i \mid i < \alpha \rangle$  of limit models of cardinality  $\mu$  with  $M_{i+1}$  universal over  $M_i$ , and for every  $p \in \text{gaS}(M_\alpha)$ , where  $M_\alpha = \bigcup_{i < \alpha} M_i$ , we have that there exists  $i < \alpha$  such that  $p$  does not  $\mu$ -split over  $M_i$ .

**Remark 3.** Shelah and Villaveces show that under the assumptions of Theorem 1,  $\mathcal{K}$  is  $\mu$ -superstable [14, Fact 2.1.3 and Theorem 2.2.1]. Their proof uses GCH, but in a non-essential way. At the point that they use  $2^{<\mu} = \mu$ , the replacement of choosing minimal  $\chi \leq \mu$  so that  $2^\chi > \mu$  would be sufficient.

We will see that, in fact, a slightly stronger form of  $\mu$ -superstability follows from categoricity. This stronger form of  $\mu$ -superstability is Definition 4 with the additional condition of  $\mu$ -symmetry. The property of  $\mu$ -symmetry was introduced in [18] and used to prove the uniqueness of limit models assuming the amalgamation property [20, 21]. Here, we will adapt these proofs to the setting of [14] where the full amalgamation property is not assumed.

Before moving to the proof of Theorem 1, we recall a fact about directed systems. The following is implicit in the proof of Theorem III.10.1 of [16]. This fact is used to construct extensions of amalgamable towers in [16]. Key is the assumption that  $\bigcup_{i < \theta} N_i$  is an amalgamation base. Without this assumption, the direct limit may not lie in  $\mathfrak{C}$ . This was exactly the point in [16] where the additional Assumption 1 was introduced in [16] to resolve one of the issues with Shelah and Villaveces' proof of the uniqueness of limit models.

**Fact 5.** Suppose that  $\theta$  is a limit ordinal and  $\langle M_i \in \mathcal{K}_\mu \mid i < \theta \rangle$  and  $\langle f_{i,j} \mid i \leq j < \theta \rangle$  form a directed system. Assume that each  $M_i$  is an amalgamation base and that each  $f_{i,j}$  can be extended to an automorphism of  $\mathfrak{C}$ . If  $\theta$  is a limit ordinal  $< \mu^+$  and  $\langle N_i \mid i \leq \theta \rangle$  is an increasing and continuous sequence of amalgamation bases so that for every  $i < \theta$ ,  $N_i \prec_{\mathcal{K}} M_i$  and  $f_{i,i+1} \upharpoonright N_i = \text{id}_{N_i}$ , then there is a direct limit  $M^* \prec_{\mathcal{K}} \mathfrak{C}$  of the system and  $\mathcal{K}$ -embeddings  $\langle f_{i,\theta} \mid i < \theta \rangle$  so that

1. each  $f_{i,\theta}$  can be extended to an automorphism of  $\mathfrak{C}$
2.  $\bigcup_{i < \theta} N_i \preceq_{\mathcal{K}} M^*$  and
3.  $f_{i,\theta} \upharpoonright N_i = \text{id}_{N_i}$ .



### 3. Structure of the Proof of Claim 1

To prove Claim 1 we show that for every pair of limit ordinals  $\theta_1, \theta_2 < \mu^+$ , every  $(\mu, \theta_1)$ -model  $M$  over  $M_0$  can be written as a  $(\mu, \theta_2)$  over  $M_0$ . We outline the construction here, but more details on this construction can be found in [16] and [8]. The idea is to build an increasing and continuous array of models with  $(\theta_1 + 1)$ -rows and  $(\theta_2 + 1)$ -columns. The  $(\theta_1 + 1)^{st}$ -row will be constructed to be relatively full (see Definition II.6.6 of [16]) and the union of this relatively full sequence of models is a  $(\mu, \theta_2)$ -limit model. We will also construct the array so that if  $M_i^j$  is the model in the  $j^{th}$  row and  $\beta^{th}$  column of the array, then  $M_\beta^{j+1}$  will be universal over  $M_\beta^j$ . This will witness that the union of the last column of the array is a  $(\mu, \theta_2)$ -limit model. See Figure 1.

$M_0$	$M_1$	$\dots M_i$	$M_{i+1}$	$\dots M_{\theta_1}^0 = \bigcup_{k < \theta_1} M_k$
$M_0^1$	$M_1^1$	$\dots M_\beta^1$	$M_{\beta+1}^1$	$\dots M_{\theta_1}^1 = \bigcup_{\gamma < \theta_1} M_\gamma^1$
$\lambda_\approx$		$\lambda_\approx$	$\lambda_\approx$	$\lambda_\approx$
$M_0^j$	$\dots$	$M_\beta^j$	$M_{\beta+1}^j$	$\dots M_{\theta_1}^j = \bigcup_{\gamma < \theta_1} M_\gamma^j$
$\lambda_\approx$		$\lambda_\approx$	$\lambda_\approx$	$\lambda_\approx$
$M_0^{j+1}$	$\dots$	$M_\beta^{j+1}$	$M_{\beta+1}^{j+1}$	$\dots M_{\theta_1}^{j+1} = \bigcup_{\gamma < \theta_1} M_\gamma^{j+1}$
$\vdots$	$\dots$	$\vdots$	$\vdots$	$\bigcup_{\gamma < \theta_1, i < \theta_2} M_\gamma^i = M_{\theta_1}^{\theta_2}$

Continuous relatively full tower of length  $\theta_1 + 1$

Figure 1: The array of models demonstrating a  $(\mu, \theta_1)$ -limit model which is also a  $(\mu, \theta_2)$ -limit model. The notation  $M \prec^u N$  represents the statement that  $M$  is universal over  $N$ .

We will view each row of the array as a tower. A *tower* is a sequence of length  $\alpha$  of amalgamation bases (specifically limit models), denoted by

$\bar{M} = \langle M_i \in \mathcal{K}_\mu \mid i < \alpha \rangle$ , along with a sequence of designated elements  $\bar{a} = \langle a_i \in M_{i+1} \setminus M_i \mid i + 1 < \alpha \rangle$ , and a sequence of designated submodels  $\bar{N} = \langle N_i \mid i + 1 < \alpha \rangle$  for which  $M_i \prec_{\mathcal{K}} M_{i+1}$ ,  $\text{ga-tp}(a_i/M_i)$  does not  $\mu$ -split over  $N_i$ , and  $M_i$  is universal over  $N_i$  (see Definition I.5.1 of [16]). The class of all towers indexed by  $\alpha$  containing models of cardinality  $\mu$  is denoted by  $\mathcal{K}_{\mu,\alpha}^*$ . When working with towers, we will use the notation  $\mathcal{T} = (\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu,\alpha}^*$  for towers of length  $\alpha$  and other abbreviations from [18] such as  $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \beta \in \mathcal{K}_{\mu,\beta}^*$  for the restriction of the tower  $(\bar{M}, \bar{a}, \bar{N})$  to index set  $\beta$ .

Notice that the sequence  $\bar{M}$  in the definition of a tower is not required to be continuous. In fact, many times we will not have continuous towers. It is exactly at the indices witnessing discontinuity that we might have a model that is not an amalgamation base over which we will need to amalgamate two extensions. Also for  $\alpha$  a limit ordinal, a continuous tower  $\mathcal{T} \in \mathcal{K}_{\mu,\alpha}^*$  may still cause us issues if the top of the tower,  $\bigcup_{i < \alpha} M_i$ , is not an amalgamation base. To avoid these problems we will restrict ourselves to nice or amalgamable towers. A tower  $\mathcal{T} \in \mathcal{K}_{\mu,\alpha}^*$  is *nice* if for every limit  $\beta < \alpha$ ,  $\bigcup_{j < \beta} M_j$  is an amalgamation base. A tower  $\mathcal{T} \in \mathcal{K}_{\mu,\alpha}^*$  is *amalgamable* if it is nice and  $\bigcup_{\gamma < \alpha} M_\gamma$  is an amalgamation base. Trivially, under the assumption that limit models are amalgamation bases, continuous towers are nice, but they may not be amalgamable. Also notice that under Assumption 2, all towers are nice and amalgamable.

To make sure that in a given column the model in the  $(i + 1)^{\text{st}}$ -row is universal over the model in the  $i^{\text{th}}$ -row, we consider the following definition of tower extensions:

**Definition 5 (Definition 3.6.3 of [14]).** For towers  $(\bar{M}, \bar{a}, \bar{N})$  and  $(\bar{M}', \bar{a}', \bar{N}')$  in  $\mathcal{K}_{\mu,\alpha}^*$ , we say

$$(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}', \bar{N}')$$

if  $\bar{a} = \bar{a}'$ ,  $\bar{N} = \bar{N}'$ ,  $M_\beta \preceq_{\mathcal{K}} M'_\beta$ , and whenever  $M'_\beta$  is a proper extension of  $M_\beta$ , then  $M'_\beta$  is universal over  $M_\beta$ . If for each  $\beta < \alpha$ ,  $M'_\beta$  is universal over  $M_\beta$  we will write  $(\bar{M}, \bar{a}, \bar{N}) < (\bar{M}', \bar{a}', \bar{N}')$ . We say that  $\mathcal{K}_{\mu,\alpha}^*$  has the *extension property* if every  $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu,\alpha}^*$  has a  $<$ -extension in  $\mathcal{K}_{\mu,\alpha}^*$ .

In [15], we notice that in order to get the extension property for towers, the argument outlined in [14] did not seem to converge, but that a direct limit construction was sufficient. In order to carry out the direct limit construction, however, we need to restrict ourselves to amalgamable towers [16].

**Fact 6** (Corollary III.10.6 of [16]). *Under Assumption 2 and the context of Theorem 1, for every amalgamable  $\mathcal{T} \in \mathcal{K}_{\mu,\alpha}^*$  there exists  $\mathcal{T}' \in \mathcal{K}_{\mu,\alpha}^*$  so that  $\mathcal{T} < \mathcal{T}'$ .*

Assumption 2 will give us the extension property for towers, but in order to complete the construction depicted in Figure 1 we will need to produce continuous extensions. In particular we will need that the lower model in the figure  $M_{\theta_1}^{\theta_2}$  is the union of the last row of the tower. To get continuous extensions we will look at reduced towers.

**Definition 6.** A tower  $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu,\alpha}^*$  is said to be *reduced* provided that for every  $(\bar{M}', \bar{a}, \bar{N}) \in \mathcal{K}_{\mu,\alpha}^*$  with  $(\bar{M}, \bar{a}, \bar{N}) \leq (\bar{M}', \bar{a}, \bar{N})$  we have that for every  $\beta < \alpha$ ,

$$(*)_{\beta} \quad M'_{\beta} \cap \bigcup_{\gamma < \alpha} M_{\gamma} = M_{\beta}.$$

Once we have the extension property for towers (Fact 6) we are able to produce reduced towers using Fact 5:

**Fact 7** (Fact III.11.3 of [16]). *Under the context of Theorem 1, for every amalgamable  $\mathcal{T} \in \mathcal{K}_{\mu,\alpha}^*$  in  $\mathfrak{C}$  there exists  $\mathcal{T}' \in \mathcal{K}_{\mu,\alpha}^*$  a reduced extension of  $\mathcal{T}$  in  $\mathfrak{C}$ .*

**Fact 8** (Lemma III.11.5 of [16]). *Under Assumption 2 and the context of Theorem 1 if  $\mathcal{T} \in \mathcal{K}_{\mu,\alpha}^*$  is reduced, then for every  $\beta < \alpha$ ,  $\mathcal{T} \restriction \beta$  is reduced.*

**Fact 9** (Theorem III.11.2 of [16]). *Under Assumption 2 and the context of Theorem 1 for  $\theta$  a limit ordinal  $< \mu^+$ , if  $\langle \mathcal{T}^i \in \mathcal{K}_{\mu,\alpha}^* \mid i < \theta \rangle$  is an  $<$ -increasing chain of continuous and reduced towers, then the union of this chain of towers is a continuous and reduced tower in  $\mathcal{K}_{\mu,\alpha}^*$ .*

Reduced towers are important because they can be shown to be continuous. However, one of the gaps in [14] was in the proof that reduced towers are continuous. This was resolved in [17] for towers in  $\mathcal{K}_{\mu,\alpha}^*$  if one assumes that  $\mathcal{K}$  is categorical in  $\mu^+$ . Later fixes appear in [8] and [21] where one assumes the amalgamation property and additional model-theoretic assumptions. In this paper we show the approach in [21] can be applied in our context with limited amalgamation. Underlying the fix in [21] is the additional assumption of  $\mu$ -symmetry. We restate the definition here introducing the nuance of amalgamation bases:

**Definition 7 (Definition 3 of [18]).** We say that an abstract elementary class exhibits  $\mu$ -*symmetry* if whenever models  $M, M_0, N \in \mathcal{K}_\mu$  and elements  $a$  and  $b$  satisfy the conditions 1-4 below, then there exists  $M^b$  a limit model over  $M_0$ , containing  $b$ , so that  $\text{ga-tp}(a/M^b)$  does not  $\mu$ -split over  $N$ .

1.  $M$  is an amalgamation base and universal over  $M_0$  and  $M_0$  is a limit model over  $N$ .
2.  $a \in M \setminus M_0$ .
3.  $\text{ga-tp}(a/M_0)$  is non-algebraic and does not  $\mu$ -split over  $N$ .
4.  $\text{ga-tp}(b/M)$  is non-algebraic and does not  $\mu$ -split over  $M_0$ .

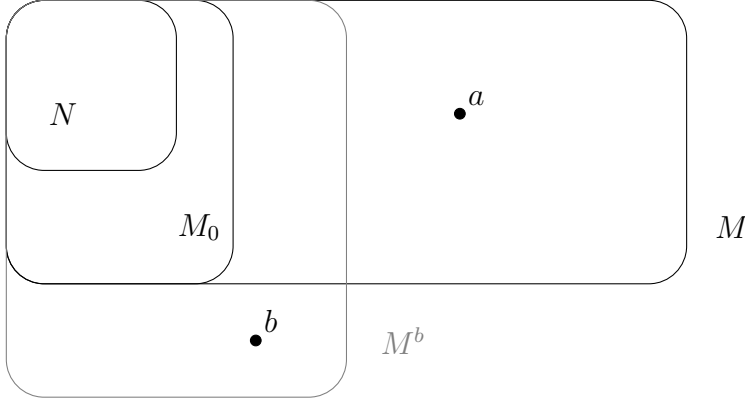


Figure 2: A diagram of the models and elements in the definition of symmetry. We assume the type  $\text{ga-tp}(b/M)$  does not  $\mu$ -split over  $M_0$  and  $\text{ga-tp}(a/M_0)$  does not  $\mu$ -split over  $N$ . Symmetry implies the existence of  $M^b$  a limit model over  $M_0$  containing  $b$  so that  $\text{ga-tp}(a/M^b)$  does not  $\mu$ -split over  $N$ .

In [18] under the assumption of the amalgamation property, this notion is shown to be equivalent to the statement that reduced towers are continuous – the gap in the proof of Theorem 3.1.15 of [14] that is acknowledged and partially, but not completely, resolved in the errata [17]. VanDieren and Vasey show that for classes that satisfy the full amalgamation property,  $\lambda$  categoricity implies  $\mu$ -symmetry for  $\mu$  satisfying  $\text{LS}(\mathcal{K}) \leq \mu < \text{cf}(\lambda)$  [21, Corollary 5.2]. In section 5, we verify that the arguments from [18] and [21] can be carried out in this context under Assumption 1, thereby fully resolving the problem described in [17]. This will show the implication ①  $\Rightarrow$  ② of Theorem 1.

#### 4. Limit and Saturated Models

In this section we verify some basic facts about saturated models in the context of Theorem 1 where only a limited amount of amalgamation is assumed. In this section we make the following assumptions which follow from the assumptions of Theorem 1:

**Hypothesis 1.** *We assume that  $\mathcal{K}$  is an abstract elementary class satisfying the following conditions for a fixed  $\kappa$  with  $\text{LS}(\mathcal{K}) \leq \kappa < \lambda$ :*

1. *Density of amalgamation bases of cardinality  $\kappa$  and  $\kappa^+$ .*
2. *Limit models of cardinality  $\kappa$  and  $\kappa^+$  are amalgamation bases.*
3. *For  $\chi = \kappa$  and  $\kappa^+$ , for every limit ordinal  $\theta < \chi^+$  and every amalgamation base  $N \in \mathcal{K}_\chi$  there exists  $M \in \mathcal{K}$  a  $(\chi, \theta)$ -limit model extending  $N$ .*

Because we do not have the full amalgamation property, it may be the case that there are two non-isomorphic Galois-saturated models of cardinality  $\kappa^+$  in our context. For instance we might have a Galois-saturated model of cardinality  $\kappa^+$  that is trivially saturated by way of having no or few submodels of cardinality  $\kappa$  that are amalgamation bases. Alternatively, we might have two saturated models: one which is an amalgamation base and one which is not. Fortunately we can avoid these kinds of anomalies in our proofs in later sections by restricting ourselves to saturated models which are dense with amalgamation bases.

**Definition 8.** A model  $M$  of cardinality  $> \kappa$  is said to be *dense with  $\kappa$ -amalgamation bases* if for every  $N \prec_{\mathcal{K}} M$  of cardinality  $\kappa$  there exists an amalgamation base  $N' \in \mathcal{K}_\kappa$  for which  $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M$ .

**Lemma 1.** *Suppose that  $M$  is a saturated model of cardinality  $\kappa^+$  that is dense with  $\kappa$ -amalgamation bases. Then  $M$  is universal over  $N$  for every amalgamation base  $N \prec_{\mathcal{K}} M$  of cardinality  $\kappa$ .*

Notice that we do not require that  $M$  be an amalgamation base at this stage; however, later, in Corollary 2 this is established.

*Proof.* The proof is an adaptation of the proof that saturated models are model homogeneous which assumes the full amalgamation property [6, Theorem 2.12]. Let  $M^*$  be a  $(\kappa, \kappa^+)$ -limit model extending  $M$  which is also universal over  $N$ . We will use  $M^*$  as a replacement for a monster model.

Fix  $N'$  a model of cardinality  $\kappa$  so that  $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M^*$ . Let  $\langle a_i \mid i < \kappa \rangle$  be an enumeration of  $N' \setminus N$ . By induction on  $i < \kappa$  we will define increasing and continuous sequences of models  $\langle N'_i \mid i < \kappa \rangle$  and  $\langle N_i \mid i < \kappa \rangle$  and mappings  $\langle f_i \mid i < \kappa \rangle$  and  $\langle f'_i \mid i < \kappa \rangle$  so that the following properties are satisfied:

1.  $N_i$  is a model of cardinality  $\kappa$  (note that we do not require  $N_i$  to be an amalgamation base.)
2.  $N'_i$  is an amalgamation base of cardinality  $\kappa$ .
3.  $N_i \prec_{\mathcal{K}} N'_i \prec_{\mathcal{K}} M^*$ .
4.  $N_0 = N$  and  $N'_0 = N'$ .
5.  $a_i \in N_{i+1}$ .
6. either  $N'_{j+1} = N'_j$  or  $N'_{j+1}$  is universal over  $N'_j$
7.  $f_i : N_i \rightarrow M$  with  $f_0 = id_N$ .
8.  $f'_i : N'_i \rightarrow M^*$  with  $f_i \subseteq f'_i$ .

Clearly this construction is sufficient since  $\bigcup_{i < \kappa} f_i \upharpoonright N'$  is as required.

The only issue that needs to be checked at limit stages is that  $N'_j$  is an amalgamation base, but this is guaranteed by conditions 2 and 6 of the construction and Hypothesis 1.

Let us consider the successor stage:  $i = j + 1$ . Suppose that  $f_j, f'_j, N_j$ , and  $N'_j$  have been defined. If  $a_j \in N_j$ , let  $N_{j+1} := N_j$ ,  $N'_{j+1} := N'_j$ ,  $f_{j+1} := f_j$ , and  $f'_{j+1} := f'_j$ . So suppose that  $a_j \notin N_j$ . Let  $M_j := f_j[N_j]$  and  $M'_j := f'_j[N'_j]$ . Notice that the diagram below commutes:

$$\begin{array}{ccc} N'_j & \xrightarrow{f'_j} & M'_j \\ \text{id} \uparrow & & \uparrow \text{id} \\ N_j & \xrightarrow{f_j} & M_j \end{array}$$

Since  $a_j \in N'_j \setminus N_j$ ,  $f'_j(a_j) \in M'_j \setminus M_j$ . There are two cases to consider:  $f'_j(a_j) \in M$  and  $f'_j(a_j) \notin M$ . If  $f'_j(a_j) \in M$ , since  $N'_j$  is an amalgamation base, we can find  $\bar{f}_j$  an automorphism of  $M^*$  extending  $f'_j$ . Let  $M_{j+1}$  be a submodel of  $M$  of cardinality  $\kappa$  extending  $M_j$  and  $f'_j(a_j)$ . Let  $N_{j+1} := \bar{f}_j^{-1}[M_j]$ . Then let  $N'_{j+1}$  be an amalgamation base of cardinality  $\kappa$  which is a universal extension over  $N'_j$ , contains  $N_{j+1}$ , and lies inside  $M^*$ . Then  $f_{j+1} := \bar{f}_j \upharpoonright N_{j+1}$  and  $f'_{j+1} := \bar{f}_j \upharpoonright N'_{j+1}$  are as required.

For the other case suppose that  $f'_j(a_j) \notin M$ . As before set  $M_j := f_j[N_j]$ . Since  $M$  is dense with amalgamation bases there exists  $\hat{M}_j \prec_\kappa M$  extending  $M_j$  which is an amalgamation base of cardinality  $\kappa$ . We can then consider the non-algebraic type,  $p := \text{ga-tp}(f'_j(a_j)/\hat{M}_j)$ . Because  $M$  is saturated there exists  $b \in M$  realizing  $p$ . Let  $\hat{M}^b$  be an amalgamation base of cardinality  $\kappa$  inside  $M$  containing  $b$  and extending  $\hat{M}_j$ . By the definition of equality of types, we can find  $h \in \text{Aut}_{\hat{M}_j} M^*$  so that  $h(b) = a$  and the following diagram commutes:

$$\begin{array}{ccc} M'_j & \xrightarrow{id} & M^* \\ id \uparrow & & \uparrow h \\ \hat{M}_j & \xrightarrow{id} & \hat{M}^b \end{array}$$

We can replace  $M^*$  in the diagram with some submodel  $\hat{M}$  of cardinality  $\kappa$  containing  $h[\hat{M}^b]$  and universal over  $M'_j$ . This is possible since  $M'_j$  is isomorphic to  $N'_j$  which was chosen to be an amalgamation base. Then gluing this diagram together with the previous diagram gives us

$$\begin{array}{ccccccc} N'_j & \xrightarrow{f'_j} & M'_j & \xrightarrow{id} & \hat{M} & & \\ id \uparrow & & id \uparrow & & \uparrow h & & \\ N_j & \xrightarrow{f_j} & M_j & \xrightarrow{id} & \hat{M}^b & \xrightarrow{id} & M \end{array}$$

Let  $N'_{j+1} := \bar{f}_j^{-1}[\hat{M}]$  and set  $N_{j+1} := \bar{f}_j^{-1}(h[\hat{M}^b])$ .

$$\begin{array}{ccccc}
& & N'_{j+1} & & \\
& & \uparrow \text{id} & \searrow \bar{f}_j & \\
N'_j & \xrightarrow{f'_j} & M'_j & \xrightarrow{\text{id}} & \hat{M} \\
\uparrow \text{id} & & \uparrow \text{id} & & \uparrow h \\
N_j & \xrightarrow{f_j} & M_j & \xrightarrow{\text{id}} & \hat{M}^b \xrightarrow{\text{id}} M
\end{array}$$

Then  $f'_{j+1} := \bar{f}_j \upharpoonright N'_{j+1}$  and  $f_{j+1} := (h^{-1} \circ \bar{f}_j) \upharpoonright N_{j+1}$  are as required.  $\square$

Notice that  $(\kappa^+, \kappa^+)$ -limit models and  $(\kappa, \kappa^+)$ -limit models are isomorphic:

**Proposition 1.** *If  $M$  is a  $(\kappa, \kappa^+)$ -limit model over  $N$  and  $M'$  is a  $(\kappa^+, \kappa^+)$ -limit model over some  $M'_0$  containing  $N$ , then  $M$  and  $M'$  are isomorphic over  $N$ .*

*Proof.* Let  $\langle M_i \in \mathcal{K}_\kappa \mid i < \kappa^+ \rangle$  witness that  $M$  is a  $(\kappa, \kappa^+)$ -limit model with  $N = M_0$  and let  $\langle M'_i \in \mathcal{K}_{\kappa^+} \mid i < \kappa^+ \rangle$  witness that  $M'$  is a  $(\kappa^+, \kappa^+)$ -limit model with  $N \prec_\kappa M'_0$ . Fix  $\langle a'_i \mid i < \kappa^+ \rangle$  an enumeration of  $M'$ .

Since the models  $M_i$  in the resolution of  $M$  are all amalgamation bases, we are able to carry out the standard construction of an isomorphism  $f : M \cong M'$  by an increasing and continuous sequence of partial mappings  $f_i : M_i \rightarrow M'$  so that  $f_0$  is the identity mapping and  $a'_i \in f_{i+1}[M_{i+1}]$ .  $\square$

Proposition 1 along with the following corollaries are used in the proof of ④  $\Rightarrow$  ① of Theorem 1. A subtlety here is that the standard proofs of the uniqueness of saturated models require that the models are dense with amalgamation bases.

**Corollary 2.** *If  $M$  is a saturated model of cardinality  $\kappa^+$  that is dense with  $\kappa$ -amalgamation bases, then  $M$  is a  $(\kappa, \kappa^+)$ -limit model.*

*Proof.* Similar to the proof of Proposition 1.  $\square$

Furthermore, we will need to show that the saturated model that we construct is in fact an amalgamation base. This follows from Proposition 1, Corollary 2, and Fact 2.



**Corollary 3.** *If  $M$  is a saturated model of cardinality  $\kappa^+$  that is dense with  $\kappa$ -amalgamation bases, then  $M$  is an amalgamation base.*

How might we construct models that are dense with amalgamation bases? First notice that  $(\kappa, \kappa^+)$ -limits are trivially dense with amalgamation bases of cardinality  $\kappa$ . Thus by Proposition 1,  $(\kappa^+, \kappa^+)$ -limit models are also dense with amalgamation bases of cardinality  $\kappa$ . This allows us to show that any limit model is dense with amalgamation bases:

**Lemma 2.** *For  $\theta$  a limit ordinal  $< \kappa^{++}$ , if  $M$  is a  $(\kappa^+, \theta)$ -limit model, then  $M$  is dense with amalgamation bases of cardinality  $\kappa$ .*

*Proof.* Let  $M$  be a  $(\kappa^+, \theta)$ -limit model. By the uniqueness of limit models of the same cofinality we may assume that  $M = \bigcup_{i < \theta} M_i$  where  $\langle M_i \mid i < \theta \rangle$  is an increasing and continuous sequence of amalgamation bases of cardinality  $\kappa^+$  so that  $M_{i+1}$  is a  $(\kappa^+, \kappa^+)$ -limit model over  $M_i$ . Then by Proposition 1, we know that each for successor  $i$ ,  $M_i$  can be viewed as a  $(\kappa, \kappa^+)$ -limit model. For each successor  $i < \theta$ , let  $\langle M_i^\alpha \in \mathcal{K}_\kappa \mid \alpha < \kappa^+ \rangle$  witness that  $M_i$  is a  $(\kappa, \kappa^+)$ -limit model.

Let  $N \prec_\mathcal{K} M$  be a submodel of cardinality  $\kappa$ . We need to find an amalgamation base  $N'$  of cardinality  $\kappa$  extending  $N$  inside  $M$ . Without loss of generality, by renumbering if necessary, we may assume that  $M_i^0 \supseteq N \cap M_i$ .

Define by induction on  $i < \theta$  an increasing and continuous sequence  $\langle N'_i \mid i < \theta \rangle$  of amalgamation bases of cardinality  $\kappa$  so that  $N'_{i+1}$  is universal over  $N'_i$ ,  $N'_i \prec_\mathcal{K} M_i$ , and  $N \cap M_i \subseteq N'_i$ . Let  $N'_0 := M_0^0$ . At limit stages  $i$ , set  $N'_i := \bigcup_{j < i} N'_j$ . Notice  $N'_i$  is a limit model by our inductive construction. And, hence, it is an amalgamation base. Now for the successor stage of the construction  $i = j + 1$ , assume that  $N'_j$  has been defined. Since  $N'_j$  has cardinality  $\kappa$ , we know that there exists  $\alpha < \kappa^+$  so that  $N'_j \prec_\mathcal{K} M_{j+1}^\alpha$ . Take  $N'_{j+1} := M_{j+1}^{\alpha+1}$ .

Notice that  $N' := \bigcup_{i < \theta} N'_i$  is a  $(\kappa, \theta)$ -limit model inside  $M$  and extends  $N$ . Since limit models are amalgamation bases, we are done.  $\square$

The following will also be used in the proof of  $\textcircled{4} \Rightarrow \textcircled{1}$  of Theorem 1.

**Lemma 3.** *Suppose that  $\theta$  is a limit ordinal  $< \kappa^{++}$ . If  $\langle M_i \mid i < \theta \rangle$  is an increasing and continuous sequence of saturated models of cardinality  $\kappa^+$  and each is dense with  $\kappa$  amalgamation bases, then  $M := \bigcup_{i < \theta} M_i$  is dense with  $\kappa$ -amalgamation bases.*

*Proof.* To see that  $M$  is dense with amalgamation bases, let  $N \prec_{\mathcal{K}} M$  have cardinality  $\kappa$ . If there exists  $i < \theta$  so that  $N \prec_{\mathcal{K}} M_i$  then we are done since by our assumption,  $M_i$  is dense with  $\kappa$  amalgamation bases so there is  $N' \prec_{\mathcal{K}} M_i \prec_{\mathcal{K}} M$  an amalgamation base of cardinality  $\kappa$  extending  $N$  as required.

So suppose that for each  $i < \theta$ ,  $N \cap M_i \neq N$ . Because each  $M_i$  is saturated and dense with amalgamation bases, by Corollary 2 each  $M_i$  is a  $(\kappa, \kappa^+)$ -limit model. This allows us to construct an increasing and continuous sequence of amalgamation bases of cardinality  $\kappa$ ,  $\langle N_i \mid i < \theta \rangle$ , so that  $N \cap M_i \prec_{\mathcal{K}} N_i \prec_{\mathcal{K}} M_i$  and  $N_{i+1}$  is universal over  $N_i$ . Notice that  $\bigcup_{i < \theta} N_i$  lies in  $M$ , extends  $N$ , and is a limit model and hence an amalgamation base.  $\square$

Note that in Lemma 3 we cannot conclude outright that  $\bigcup_{i < \theta} M_i$  is also saturated without assuming some superstability.

## 5. Symmetry and reduced towers

In this section we discuss the connection between the uniqueness of limit models and  $\mu$ -symmetry. This is used to prove ①  $\Rightarrow$  ② of Theorem 1.

For this section we make the following hypothesis:

**Hypothesis 2.** *We assume that  $\mathcal{K}$  is an abstract elementary class satisfying the following conditions for every  $\kappa$  with  $\text{LS}(\mathcal{K}) \leq \kappa < \lambda$ ,*

1. *Limit models of cardinality  $\kappa$  are amalgamation bases.*
2. *For every limit ordinal  $\theta < \kappa^+$  and every amalgamation base  $N \in \mathcal{K}_{\kappa}$  there exists  $M \in \mathcal{K}$  a  $(\kappa, \theta)$ -limit model over  $N$ .*
3.  *$\mathcal{K}$  is  $\kappa$ -superstable.*
4. *The union of an increasing chain of limit models of cardinality  $\kappa$  is an amalgamation base (Assumption 2).*

We show that the arguments from [18] and [21] can be carried out without the amalgamation property, if we assume only Hypothesis 2, to prove that reduced towers are continuous in categorical classes:

**Theorem 2.** *Suppose that Hypothesis 2 holds. Suppose that  $\lambda$  and  $\mu$  are cardinals so that there exists  $0 < n < \omega$  so that  $\text{LS}(\mathcal{K}) \leq \mu < \mu^{+n} = \lambda$ . If  $\mathcal{K}$  is categorical in  $\lambda$ , then reduced towers in  $\mathcal{K}_{\mu, \alpha}^*$  are continuous if  $\alpha < \mu^+$ .*

*Proof.* When we take  $n = 1$ , Theorem 2 reduces to Theorem 2 of [17]. The case  $n > 1$  of Theorem 2 is proved by first showing that  $\mu$ -symmetry implies that reduced towers are continuous (Theorem 3) and then deriving  $\mu$ -symmetry from categoricity in  $\lambda = \mu^{+n}$  (Theorem 5).  $\square$

The remainder of the section is dedicated to prove the two theorems referenced in the proof of Theorem 2. In order to prove Theorem 5, we need the converse of Theorem 3. We begin by establishing the equivalence of  $\mu$ -symmetry and the statement that reduced towers of cardinality  $\mu$  are continuous (Theorem 3 and its converse Theorem 4). Then we finish the section by proving that  $\mu$ -symmetry can be derived from categoricity in  $\mu^{+n}$  for some  $0 < n < \omega$  (Theorem 5).

**Theorem 3** (Adaptation of Theorem 5 of [18]). *Suppose that Hypothesis 2 holds and that Assumption 2 holds. If  $\mathcal{K}$  has symmetry for non- $\mu$ -splitting, then for  $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, \alpha}^*$  a reduced tower, we can conclude that  $\bar{M}$  is a continuous sequence (i.e. for every limit ordinal  $\beta < \alpha$ , we have  $M_\beta = \bigcup_{\gamma < \beta} M_\gamma$ ).*

*Proof.* Suppose  $\mathcal{K}$  has symmetry for non- $\mu$ -splitting, but reduced towers are not necessarily continuous. Let  $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, \alpha}^*$  be a discontinuous reduced tower in  $\mathfrak{C}$  of minimal length,  $\alpha$ . Notice that by Fact 8, we can conclude that  $\alpha = \delta + 1$  for some limit ordinal  $\delta$  and that the failure of continuity must occur at  $\delta$ . Let  $b \in M_\delta \setminus \bigcup_{\gamma < \delta} M_\gamma$  witness the discontinuity of the tower. By Assumption 2,  $\bigcup_{\gamma < \delta} M_\gamma$  must be an amalgamation base.

By the minimality of  $\alpha$  and the density of reduced towers (Fact 7 and Fact 9) we can construct a  $<$ -increasing and continuous chain of reduced, continuous towers  $\langle \mathcal{T}^i = (\bar{M}, \bar{a}, \bar{N})^i \in \mathcal{K}_{\mu, \delta}^* \mid i < \delta \rangle$  with  $(\bar{M}, \bar{a}, \bar{N})^0 := (\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$  inside  $\mathfrak{C}$ . By  $\delta$ -applications of Fact 7 inbetween successor stages of the construction we can require that for  $\beta < \delta$

$$M_\beta^{i+1} \text{ is a } (\mu, \delta)\text{-limit over } N_\beta. \quad (1)$$

Let  $M_\delta^\delta := \bigcup_{i < \delta, \beta < \delta} M_\beta^i$ . See Figure 3.

There are two cases: 1) we have  $b \in M_\delta^\delta$  and 2) we have  $b \notin M_\delta^\delta$ . If  $b \in M_\delta^\delta$ , then we will have found an extension of  $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$  containing  $b$  (namely  $(\bar{M}, \bar{a}, \bar{N})^\delta$ ) which can easily be lengthened to a discontinuous extension of the entire  $(\bar{M}, \bar{a}, \bar{N})$  tower by taking the  $\delta^{th}$  model to be some

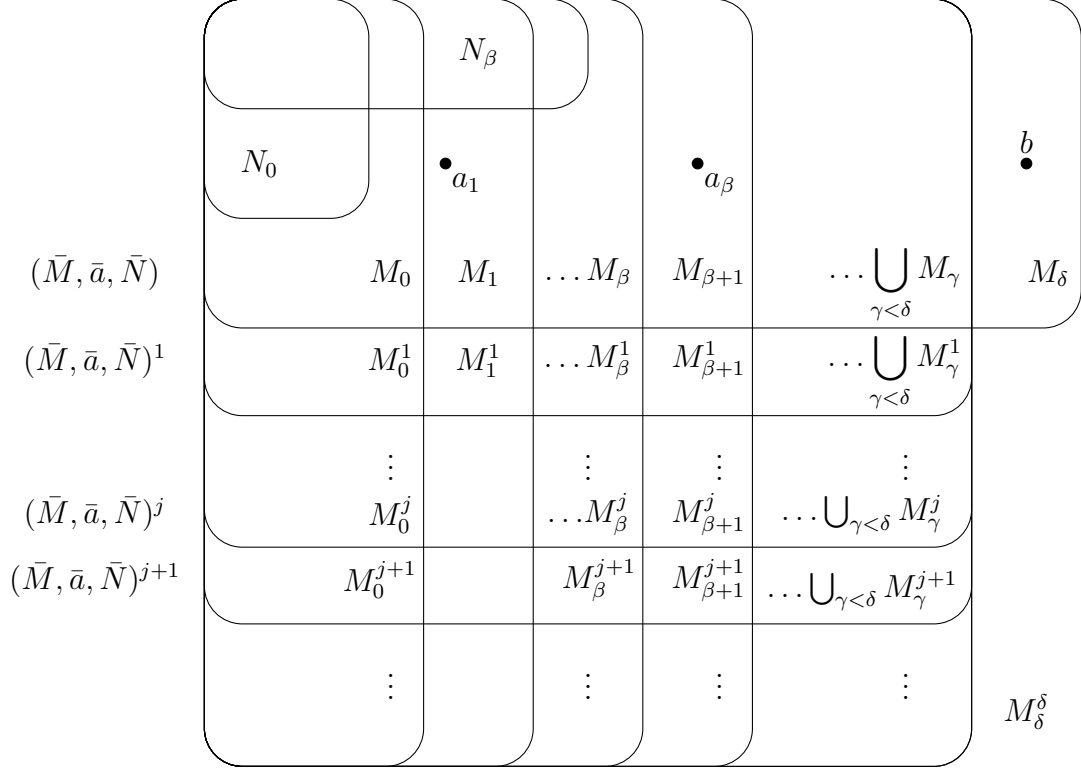


Figure 3:  $(\bar{M}, \bar{a}, \bar{N})$  and the towers  $(\bar{M}, \bar{a}, \bar{N})^j$  extending  $(\bar{M}, \bar{a}, \bar{N}) \upharpoonright \delta$  inside  $\mathfrak{C}$ .

extension of  $M_\delta^\delta$  which is also universal over  $M_\delta$ . This is possible because we have constructed  $M_\delta^\delta$  so that it lies in  $\mathfrak{C}$  along with  $M_\delta$ . This discontinuous extension of  $(\bar{M}, \bar{a}, \bar{N})$  along with  $b$  witness that  $(\bar{M}, \bar{a}, \bar{N})$  cannot be reduced.

So suppose that  $b \notin M_\delta^\delta$ . Since  $M_\delta^\delta$  is a limit model and hence an amalgamation base, we can consider the non-algebraic type  $\text{ga-tp}(b/M_\delta^\delta)$ . By the  $\mu$ -superstability assumption, there exists  $i^* < \alpha$  so that  $\text{ga-tp}(b/M_\delta^\delta)$  does not  $\mu$ -split over  $M_{i^*}^{i^*}$ . By monotonicity of non-splitting, we may assume that  $i^*$  is a successor and thus by (1),  $M_{i^*}^{i^*}$  is a  $(\mu, \delta)$ -limit over  $N_{i^*}$ . Now, referring to the Figure 2, apply symmetry to  $a_{i^*}$  standing in for  $a$ ,  $M_{i^*}^{i^*}$  representing  $M_0$ ,  $N_{i^*}$  as  $N$ ,  $M_\delta^\delta$  as  $M$ , and  $b$  as itself. We can conclude that there exists  $M^b$  containing  $b$ , a limit model over  $M_{i^*}^{i^*}$ , for which  $\text{tp}(a_{i^*}/M^b)$  does not  $\mu$ -split over  $N_{i^*}$ .

Our next step is to consider the tower formed by the diagonal elements in Figure 3. In particular let  $\mathcal{T}^{diag}$  be the tower in  $\mathcal{K}_{\mu, \delta}^*$  extending  $\mathcal{T} \upharpoonright \delta$  whose

models are  $M_i^i$  for each  $i < \delta$ .

Define the tower  $\mathcal{T}^b \in \mathcal{K}_{\mu, i^*+2}^*$  by the sequences  $\bar{a} \restriction (i^* + 1)$ ,  $\bar{N} \restriction (i^* + 1)$  and  $\bar{M}'$  with  $M_j' := M_j^j$  for  $j \leq i^*$  and  $M_{i^*+1}' := M^b$ . Notice that  $\mathcal{T}^b$  is an extension of  $\mathcal{T}^{diag} \restriction (i^* + 2)$  containing  $b$ . We will explain how we can use this tower to find a tower  $\mathring{\mathcal{T}}^\delta \in \mathcal{K}_{\mu, \delta}^*$  extending  $\mathcal{T}^{diag}$  with  $b \in \bigcup_{j < \delta} \mathring{M}_j^\delta$ . This will be enough to contradict our assumption that  $\mathcal{T}$  was reduced.

We define  $\langle \mathring{\mathcal{T}}^j, f_{j,k} \mid i^* + 2 \leq j \leq k \leq \delta \rangle$  a directed system of towers so that for  $j \geq i^* + 2$

1.  $\mathring{\mathcal{T}}^{i^*+2} = \mathcal{T}^b$
2. for  $j \leq \delta$ ,  $\mathring{\mathcal{T}}^j \in \mathcal{K}_{\mu, j}^*$  and lies in  $\mathfrak{C}$
3.  $\mathcal{T}^{diag} \restriction j \leq \mathring{\mathcal{T}}^j$  for  $j \leq \delta$
4.  $f_{j,k}(\mathring{\mathcal{T}}^j) \leq \mathring{\mathcal{T}}^k \restriction j$  for  $j \leq k < \delta$
5.  $f_{j,k} \restriction M_j^j = id_{M_j^j}$   $j \leq k < \delta$
6.  $\mathring{M}_{j+1}^{j+1}$  is universal over  $f_{j,j+1}(\mathring{M}_j^j)$  for  $j < \delta$
7.  $b \in \mathring{M}_j^j$  for  $j \leq \delta$
8.  $\text{ga-tp}(f_{j,k}(b)/M_k^k)$  does not  $\mu$ -split over  $M_{i^*}^{i^*}$  for  $j < k < \delta$ .

We will define this directed system by induction on  $k$ , with  $i^* + 2 \leq k \leq \alpha$ . The base case  $i^* + 2$  is determined by condition 1. To cover the successor case, suppose that  $k = j + 1$ . By our choice of  $i^*$ , we have  $\text{ga-tp}(b/\bigcup_{l < \alpha} M_l^l)$  does not  $\mu$ -split over  $M_{i^*}^{i^*}$ . So in particular by monotonicity of non-splitting, we notice:

$$\text{ga-tp}(b/M_{j+1}^{j+1}) \text{ does not } \mu\text{-split over } M_{i^*}^{i^*}. \quad (2)$$

Using the definition of towers, the choice of  $i^*$ , and the fact that  $M_{j+1}^{j+1}$  was chosen to be a  $(\mu, \delta)$ -limit over  $N_{j+1}$ , we can apply symmetry to  $a_{j+1}$ ,  $M_{j+1}^{j+1}$ ,  $\bigcup_{l < \delta} M_l^l$ ,  $b$  and  $N_{j+1}$  which will yield  $M_{j+1}^b$  a limit model over  $M_{j+1}^{j+1}$  containing  $b$  so that  $\text{ga-tp}(a_{j+1}/M_{j+1}^b)$  does not  $\mu$ -split over  $N_{j+1}$  (see Figure 4).

Fix  $M'$  to be a model of cardinality  $\mu$  extending both  $\mathring{M}_j^j$  and  $M_{j+1}^{j+1}$ . Since  $M_{j+1}^b$  is a limit model over  $M_{j+1}^{j+1}$ , there exists  $f_{j,j+1} : M' \rightarrow M_{j+1}^b$  with

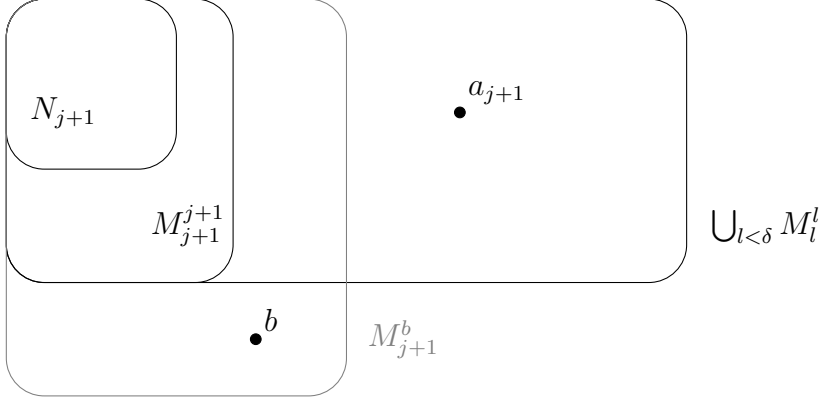


Figure 4: A diagram of the application of symmetry in the successor stage of the directed system construction in the proof of Theorem 2. We have  $\text{ga-tp}(b/\bigcup_{l < \delta} M_l^l)$  does not  $\mu$ -split over  $M_{j+1}^{j+1}$  and  $\text{ga-tp}(a_{j+1}/M_{j+1}^{j+1})$  does not  $\mu$ -split over  $N_{j+1}$ . Symmetry implies the existence of  $M^b$  a limit model over  $M_{j+1}^{j+1}$  so that  $\text{ga-tp}(a_{j+1}/M^b)$  does not  $\mu$ -split over  $N_{j+1}$ .

$f_{j,j+1} = \text{id}_{M_{j+1}^{j+1}}$  so that  $M_{j+1}^b$  is also universal over  $f_{j,j+1}(\mathring{M}_j^j)$ . Notice that condition 8 of the construction is satisfied because of (2), invariance, and our choice of  $f_{j,j+1} \upharpoonright M_{j+1}^{j+1} = \text{id}$ . Therefore, it is easy to check that  $\hat{\mathcal{T}}^{j+1}$  defined by the models  $\mathring{M}_l^{j+1} := f_{j,j+1}(\mathring{M}_l^j)$  for  $l \leq j$  and  $\mathring{M}_{j+1}^{j+1} := M_{j+1}^b$  are as required. Then the rest of the directed system can be defined by the induction hypothesis and the mappings  $f_{l,j+1} := f_{l,j} \circ f_{j,j+1}$  for  $i^* + 2 \leq l < j$ .

Now consider the limit stage  $k$  of the construction. First, let  $\hat{\mathcal{T}}^k$  and  $\langle \mathring{f}_{j,k} \mid i^* + 2 \leq j < k \rangle$  be a direct limit of the system defined so far. We use the notation since these are only approximations to the tower and mappings that we are looking for. We will have to take some care to find a direct limit that contains  $b$  in order to satisfy Condition 7 of the construction. By Assumption 2, our induction hypothesis, and Fact 5, we may choose this direct limit to lie in  $\mathfrak{C}$  so that for all  $j < k$

$$\mathring{f}_{j,k} \upharpoonright M_j^j = \text{id}_{M_j^j}.$$

Consequently  $\mathring{M}_j^\alpha := \mathring{f}_{j,k}(\mathring{M}_j^j)$  is universal over  $M_j^j$ , and  $\bigcup_{j < k} \mathring{M}_j^k$  is a limit model witnessed by condition 6 of the construction. Additionally, because  $\mathcal{T}^{diag} \upharpoonright k$  is continuous, the tower  $\hat{\mathcal{T}}^k$  composed of the models  $\mathring{M}_j^k$ , extends  $\mathcal{T}^{diag} \upharpoonright k$ .

We will next show that for every  $j < k$ ,

$$\text{ga-tp}(\dot{f}_{i^*+2,k}(b)/M_j^j) \text{ does not } \mu\text{-split over } M_{i^*}^{i^*}. \quad (3)$$

To see this, recall that for every  $j < k$ , by the definition of a direct limit,  $\dot{f}_{i^*+2,k}(b) = \dot{f}_{j,k}(f_{i^*+2,j}(b))$ . By condition 8 of the construction, we know

$$\text{ga-tp}(f_{i^*+2,j}(b)/M_j^j) \text{ does not } \mu\text{-split over } M_{i^*}^{i^*}.$$

Applying  $\dot{f}_{j,k}$  to this implies  $\text{ga-tp}(\dot{f}_{i^*+2,k}(b)/M_j^j)$  does not  $\mu$ -split over  $M_{i^*}^{i^*}$ , establishing (3).

Because  $M_{j+1}^{j+1}$  is universal over  $M_j^j$  by construction, we can apply our assumption of  $\mu$ -superstability to (3) yielding

$$\text{ga-tp}(\dot{f}_{i^*+2,k}(b)/\bigcup_{j < k} M_j^j) \text{ does not } \mu\text{-split over } M_{i^*}^{i^*}. \quad (4)$$

Because  $\dot{f}_{i^*+2,k}$  fixes  $M_{i^*+1}^{i^*+1}$ ,  $\text{ga-tp}(b/M_{i^*+1}^{i^*+1}) = \text{ga-tp}(\dot{f}_{i^*+2,k}(b)/M_{i^*+1}^{i^*+1})$ . We can then apply the uniqueness of non-splitting extensions to (4) to see that  $\text{ga-tp}(\dot{f}_{i^*+2,k}(b)/\bigcup_{j < k} M_j^j) = \text{ga-tp}(b/\bigcup_{j < k} M_j^j)$ . Thus we can fix  $g$  an automorphism of  $\mathfrak{C}$  fixing  $\bigcup_{j < k} M_j^j$  so that  $g(\dot{f}_{i^*+2,k}(b)) = b$ .

We will then define  $\dot{\mathcal{T}}^k$  to be the tower  $g(\dot{\mathcal{T}}^k)$  and the mappings for our directed system will be  $f_{j,k} := g \circ \dot{f}_{j,k}$  for all  $i^* + 2 \leq j < k$ . This completes the construction.

Now that we have  $\dot{\mathcal{T}}^\delta$  a tower extending  $\mathcal{T} \upharpoonright \delta$  which contains  $b$ , we are in a situation similar to the proof in case 1). To contradict that  $\mathcal{T}$  is reduced, we need only lengthen  $\dot{\mathcal{T}}^\delta$  to a discontinuous extension of the entire  $(\bar{M}, \bar{a}, \bar{N})$  tower by taking the  $\delta^{th}$  model to be some extension of  $\bigcup_{i < \delta} \dot{M}_i^i$  which is also universal over  $M_\delta$ . This is possible because all the models lie in  $\mathfrak{C}$ . This discontinuous extension of  $(\bar{M}, \bar{a}, \bar{N})$  along with  $b$  witness that  $(\bar{M}, \bar{a}, \bar{N})$  cannot be reduced. □

Next we adapt the proof of Theorem 5 of [18] to prove the converse of Theorem 3.

**Theorem 4** (Adaptation of Theorem 5 of [18]). *Suppose that Hypothesis 2 holds. If every  $(\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, \alpha}^*$  reduced tower is continuous (i.e. for every limit ordinal  $\beta < \alpha$ , we have  $M_\beta = \bigcup_{i < \beta} M_i$ ), then  $\mathcal{K}$  has symmetry for non- $\mu$ -splitting.*

*Proof.* Suppose that  $M$  is an amalgamation base and universal over  $M_0$  and that  $M_0$  is a limit model over  $N$  and so that all these models lie in  $\mathfrak{C}$ . Fix  $b$  so that the non-algebraic  $\text{ga-tp}(b/M)$  does not  $\mu$ -split over  $N$  with  $b \in \mathfrak{C}$ . Fix  $a \in M \setminus M_0$ . Without loss of generality, by monotonicity of non-splitting, we may assume that  $M$  is a limit model over  $M_0$ . Let  $\langle M_i \mid i < \delta \rangle$  witness this. We can arrange that  $M_{i+1}$  is a limit model over  $M_i$  and  $a \in M_1$ . To prove  $\mu$ -symmetry, we will find  $M^b$  a limit model over  $M_0$  containing  $b$  and extending  $N$  so that  $\text{ga-tp}(a/M^b)$  does not  $\mu$ -split over  $N$ .

We start by building a tower of length  $\delta + 1$ . We'll use the models in the sequence  $\langle M_i \mid i < \delta \rangle$  as the first part of the tower and we'll define  $M_\delta$  to be some limit model extending  $M$  containing  $b$ . We will set  $a_0 := a$  and for  $0 < i < \delta$  we can choose  $a_i \in M_{i+1} \setminus M_i$  realizing the extension of  $\text{ga-tp}(a/M_0)$  to  $M_i$  that does not  $\mu$ -split over  $N$ . Then set  $N_i := N$  for each  $i$ . Refer to the tower of length  $\delta + 1$  defined this way as  $\mathcal{T}$ .

Notice that  $\mathcal{T}$  is discontinuous at  $\delta$ ; therefore by our assumption, it is not reduced. However at this place of discontinuity,  $\bigcup_{i < \delta} M_i$  is a limit model and hence an amalgamation base. Therefore  $\mathcal{T}$  is amalgamable. By the  $\mu$ -superstability assumptions, our assumption that reduced towers are continuous, and Fact 7, we can find  $\mathcal{T}'$  in  $\mathfrak{C}$  extending  $\mathcal{T}$  that is reduced, and continuous. By the continuity of this tower, since  $b$  appears in the tower, there exists  $j < \delta$  so that  $b \in M'_j$ . Fix the minimal such  $j$  and denote it by  $j^*$ . There are two cases to consider

Case 1:  $j^* = 0$ . By definition of the ordering on towers, since  $\mathcal{T} < \mathcal{T}'$ , we know that  $\text{ga-tp}(a_0/M'_0)$  does not  $\mu$ -split over  $N$ . Thus  $M'_0$  witnesses  $\mu$ -symmetry.

Case 2:  $j^* > 0$ . By the choice of  $a_j$  and uniqueness of non-splitting extensions, we know  $\text{ga-tp}(a_0/M'_0) = \text{ga-tp}(a_{j^*}/M'_0)$ . Thus, there exists  $f \in \text{Aut}_{M_0}(\mathfrak{C})$  with  $f(a_{j^*}) = a_0$ . Since  $M_1$  is universal over  $M_0$ , we can also require that our choice of  $f$  has the property that  $f \upharpoonright M : M \rightarrow_{M_0} M_1$ . Because  $\text{ga-tp}(b/M)$  does not  $\mu$ -split over  $N$ , we know

$$\text{ga-tp}(f(b)/f(M)) = \text{ga-tp}(b/f(M)).$$

This implies there exists an automorphism  $g$  of  $\mathfrak{C}$  fixing  $f(M)$  so that  $g(f(b)) = b$ .

We claim that  $M^b := g(f(M'_{j^*}))$  is as required. First notice that  $b \in M^b$  since  $f(b) \in f(M'_{j^*})$  and  $g(f(b)) = b$ . Next we need to check that



$\text{ga-tp}(a_0/M^b)$  does not  $\mu$ -split over  $N$ . By the definition of towers,

$$\text{ga-tp}(a_{j^*}/M'_{j^*}) \text{ does not } \mu\text{-split over } N_{j^*}(=N).$$

By invariance and by our choice of  $f$  and  $g$  fixing  $N$  with  $g(f(M'_{j^*})) = M^b$ , we can conclude that

$$\text{ga-tp}(g(f(a_{j^*}))/M^b) \text{ does not } \mu\text{-split over } N.$$

By our choice of  $f$  taking  $a_{j^*}$  to  $a_0$ , we get

$$\text{ga-tp}(g(a_0)/M^b) \text{ does not } \mu\text{-split over } N. \quad (5)$$

Because  $g$  fixes  $f(M)$  and  $a_0 = f(a_{j^*}) \in f(M)$ , (5) implies that  $\text{ga-tp}(a_0/M^b)$  does not  $\mu$ -split over  $N$  as required.  $\square$

Combining Theorem 4 with Theorem 2 of [17], we conclude

**Corollary 4.** *Under Hypothesis 2, categoricity in  $\mu^+$  implies  $\mu$ -symmetry.*

*Proof.* Assumption 2 implies that all towers are nice. Theorem 2 of [17] states that all reduced nice towers of cardinality  $\mu$  are continuous provided that the class is categorical in  $\mu^+$ . Then Theorem 4 gives us  $\mu$ -symmetry.  $\square$

Now that we have symmetry in  $\lambda$  from categoricity in  $\lambda^+$  we can adapt the proof of Corollary 4.1 of [20] to transfer symmetry from  $\lambda$  down to  $\mu$  where  $\mu^{+n} = \lambda$  for some  $1 < n < \omega$  in this context in which the full amalgamation property is not assumed. To transfer symmetry even further down past a limit cardinal we will need to adapt the proof of the Theorem 1.1 of [21] which appears in an upcoming paper.

**Theorem 5.** *Under Hypothesis 2, categoricity in  $\mu^{+n}$  for some  $0 < n < \omega$  implies  $\mu$ -symmetry for all  $\mu \geq \text{LS}(\mathcal{K})$ .*

*Proof.* By Corollary 4 it is enough to show that  $\mu^+$ -symmetry implies  $\mu$ -symmetry. This is an adaptation of the proof of Theorem 0.1 of [20]. Suppose  $\mathcal{K}$  does not have symmetry for  $\mu$ -non-splitting. By Theorem 4 and Hypothesis 2,  $\mathcal{K}$  has a reduced discontinuous tower. Let  $\alpha$  be the minimal ordinal such that  $\mathcal{K}$  has a reduced discontinuous tower of length  $\alpha$ . By Fact 8, we may assume that  $\alpha = \delta + 1$  for some limit ordinal  $\delta$ . Fix  $\mathcal{T} = (\bar{M}, \bar{a}, \bar{N}) \in \mathcal{K}_{\mu, \alpha}^*$

a reduced discontinuous tower with  $b \in M_\delta \setminus \bigcup_{\beta < \delta} M_\beta$ . By Fact 7, Fact 9, and the minimality of  $\alpha$ , we can build an increasing and continuous chain of reduced, continuous towers  $\langle \mathcal{T}^i \mid i < \mu^+ \rangle$  extending  $\mathcal{T} \upharpoonright \delta$  in  $\mathfrak{C}$ .

For each  $\beta < \delta$ , set  $M_\beta^{\mu^+} := \bigcup_{i < \mu^+} M_\beta^i$ . Notice that for each  $\beta < \delta$

$$\text{ga-tp}(a_\beta/M_\beta^{\mu^+}) \text{ does not } \mu\text{-split over } N_\beta. \quad (6)$$

If  $\text{ga-tp}(a_\beta/M_\beta^{\mu^+})$  did  $\mu$ -split over  $N_\beta$ , it would be witnessed by models inside some  $M_\beta^i$ , contradicting the fact that  $\text{ga-tp}(a_\beta/M_\beta^i)$  does not  $\mu$ -split over  $N_\beta$ .

We will construct a tower in  $\mathcal{K}_{\mu^+, \delta}^*$  from  $\bar{M}^{\mu^+}$ . Notice that by construction, each  $M_\beta^{\mu^+}$  is a  $(\mu, \mu^+)$ -limit model. By Hypothesis 2.2, there is a  $(\mu^+, \mu^+)$ -limit model; so we can apply Proposition 1 to notice that each  $M_\beta^{\mu^+}$  can be represented as a  $(\mu^+, \mu^+)$ -limit model. Fix  $\langle \bar{M}_\beta^i \mid i < \mu^+ \rangle$  witnessing that  $M_\beta^{\mu^+}$  is a  $(\mu^+, \mu^+)$ -limit model. Without loss of generality we can assume that  $N_\beta \prec_{\mathcal{K}} \bar{M}_\beta^0$ . By  $\mu^+$ -superstability we know that for each  $\beta < \delta$  there is  $i(\beta) < \mu^+$  so that  $\text{ga-tp}(a_\beta/M_\beta^{\mu^+})$  does not  $\mu^+$ -split over  $\bar{M}_\beta^{i(\beta)}$ . Set  $N_\beta^{\mu^+} := \bar{M}_\beta^{i(\beta)}$ . Notice that  $(\bar{M}^{\mu^+}, \bar{a}, \bar{N}^{\mu^+})$  is a tower in  $\mathcal{K}_{\mu^+, \delta}^*$  that lies in  $\mathfrak{C}$ . Extend  $(\bar{M}^{\mu^+}, \bar{a}, \bar{N}^{\mu^+})$  to a tower  $\mathcal{T}^{\mu^+} \in \mathcal{K}_{\mu^+, \alpha}^*$  by appending to  $\bar{M}^{\mu^+}$  a  $\mu^+$ -limit model universal over  $M_\delta$  which contains  $\bigcup_{\beta < \delta} M_\beta^{\mu^+}$ . This is possible since all of these models lie in  $\mathfrak{C}$ . Since  $\mathcal{T}^{\mu^+}$  is discontinuous, by Theorem 3 and our  $\mu^+$ -symmetry assumption, we know that it is not reduced.

However, by Hypothesis 2, our  $\mu^+$ -symmetry assumption, Theorem 3 and Fact 7 imply that there exists a reduced, continuous tower  $\mathcal{T}^* \in \mathcal{K}_{\mu^+, \alpha}^*$  extending  $\mathcal{T}^{\mu^+}$  in  $\mathfrak{C}$ . By multiple applications of Fact 7, we may assume that in  $\mathcal{T}^*$  each  $M_\beta^*$  is a  $(\mu^+, \mu^+)$ -limit over  $M_\beta^{\mu^+}$ . See Fig. 5.

**Claim 2.** *For every  $\beta < \alpha$ ,  $\text{ga-tp}(a_\beta/M_\beta^*)$  does not  $\mu$ -split over  $N_\beta$ .*

*Proof.* Since  $M_\beta^*$  and  $M_\beta^{\mu^+}$  are both  $(\mu^+, \mu^+)$ -limit models over  $N_\beta^{\mu^+}$ , there exists  $f : M_\beta^* \cong_{N_\beta^{\mu^+}} M_\beta^{\mu^+}$ . Since  $\mathcal{T}^*$  is a tower extending  $\mathcal{T}^{\mu^+}$ , we know that  $\text{ga-tp}(a_\beta/M_\beta^*)$  does not  $\mu^+$ -split over  $N_\beta^{\mu^+}$ . Therefore by the definition of non-splitting, it must be the case that  $\text{ga-tp}(f(a_\beta)/M_\beta^{\mu^+}) = \text{ga-tp}(a_\beta/M_\beta^{\mu^+})$ . From this equality of types we can fix  $g \in \text{Aut}_{M_\beta^{\mu^+}}(\mathfrak{C})$  with  $g(f(a_\beta)) = a_\beta$ . An application of  $(g \circ f)^{-1}$  to (6) yields the statement of the claim.  $\square$

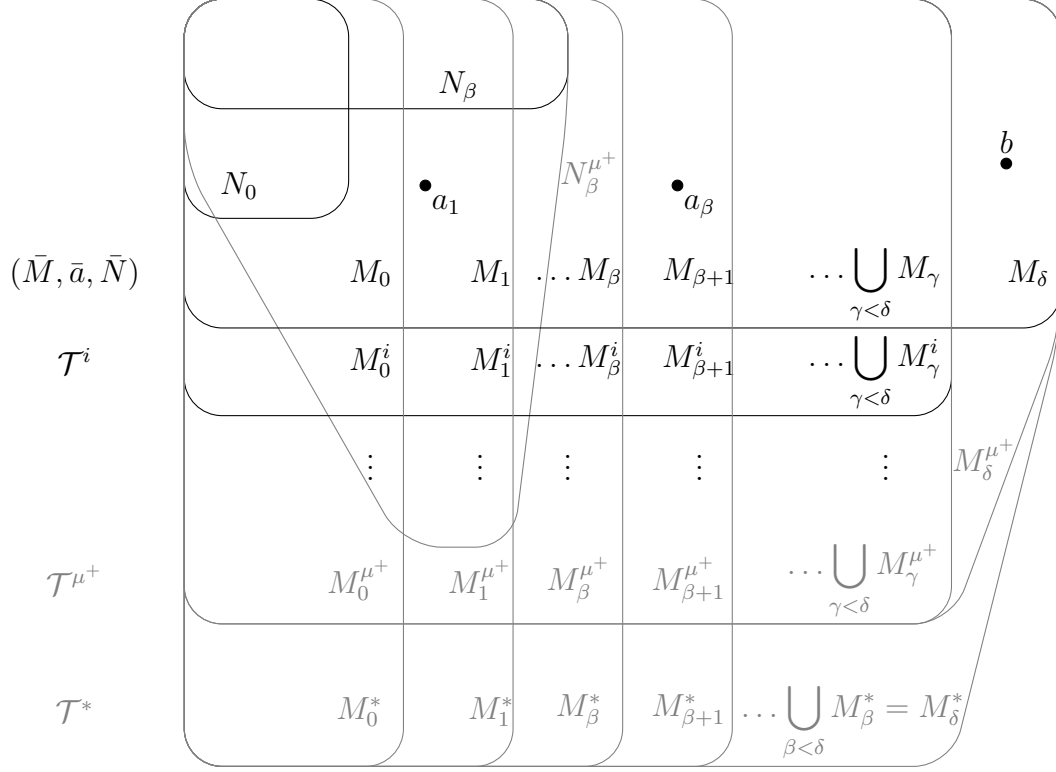


Figure 5: The towers in the proof of Theorem 5. The towers composed of models of cardinality  $\mu$  are black and the towers composed of models of cardinality  $\mu^+$  are gray.

Since  $\mathcal{T}^*$  is continuous and extends  $\mathcal{T}^{\mu^+}$  which contains  $b$ , there is  $\beta < \delta$  such that  $b \in M_\beta^*$ . Fix such a  $\beta$ .

We now will define a tower  $\mathcal{T}^b \in \mathcal{K}_{\mu, \alpha}^*$  extending  $\mathcal{T}$ . For  $\gamma < \beta$ , take  $M_\gamma^b := M_\gamma$ . For  $\gamma = \beta$ , let  $M_\beta^b$  be a  $(\mu, \mu)$ -limit model over  $M_\gamma$  inside  $M_\gamma^*$  so that  $b \in M_\beta^b$ . For  $\gamma > \beta$ , take  $M_\gamma^b$  to be a  $(\mu, \mu)$ -limit model over  $M_\gamma$  so that  $\bigcup_{\xi < \gamma} M_\xi^b \prec_{\mathcal{K}} M_\gamma^b$ . Notice that by Claim 2 and monotonicity of non-splitting, the tower  $\mathcal{T}^b$  defined as  $(\bar{M}^b, \bar{a}, \bar{N})$  is a tower extending  $\mathcal{T}$  with  $b \in (M_\beta^b \setminus M_\beta) \cap M_\alpha$ . This contradicts our assumption that  $\mathcal{T}$  was reduced.  $\square$

## 6. Proof of Theorem 1

First notice that the assumptions of Theorem 1 imply the following properties for every  $\kappa$  with  $\text{LS}(\mathcal{K}) \leq \kappa < \lambda$ :

1.  $\kappa$ -superstability [16, Facts 1.4.7 and 1.48].
2. Limit models of cardinality  $\kappa$  are amalgamation bases [14, Fact 1.3.10].
3. Density of amalgamation bases of cardinality  $\kappa$  [14, Theorem 1.2.4].
4. For every amalgamation base  $M$  of cardinality  $\kappa$  there exists  $M' \in \mathcal{K}_\mu$  a limit model over  $M$  [14, Fact 1.3.10].

*Proof of ①  $\Rightarrow$  ② of Theorem 1.* This is the content of [16] along with Theorem 2.  $\square$

*Proof of ②  $\Rightarrow$  ③.* Suppose that  $M$  is a  $(\mu, \theta)$ -limit model over  $M_0$  and  $M'$  is a  $(\mu, \theta')$ -limit model over  $M'_0$  (perhaps of no relation to  $M_0$ ). By categoricity in  $\lambda$  we may assume without loss of generality that there is  $N \in \mathcal{K}_\lambda$  so that  $M, M' \prec_\kappa N$ . By the Downward Löwenheim Skolem axiom of AECs, we can find  $M^*$  an extension of  $M$  of cardinality  $\mu$  containing  $M'$ . By the coherence axiom, we may assume that  $M' \prec_\kappa M^*$  as well. By the existence of limit models, we can assume that  $M^*$  is a  $(\mu, \theta')$ -limit model over  $M$ . Notice that  $M^*$  is also a  $(\mu, \theta')$ -limit model over  $M_0$ . By ②,  $M^*$  and  $M$  are isomorphic over  $M_0$ .

Furthermore, notice that  $M^*$  is a  $(\mu, \theta')$ -limit model over  $M'$  as well. Then we also know that  $M^*$  is a  $(\mu, \theta')$ -limit model over  $M'_0$ . By a back and forth construction  $M^*$  and  $M'$  are isomorphic over  $M'_0$ . Thus, combining this information with the previous paragraph, we conclude that  $M'$  and  $M$  are isomorphic.  $\square$

*Proof of ③  $\Rightarrow$  ④ of Theorem 1.* This argument is an adaptation of the proof of Corollary 3 of [19]. Fix  $M = \bigcup_{i < \theta} M_i$  where  $\langle M_i \in \mathcal{K}_{\kappa^+} \mid i < \theta \rangle$  is an increasing and continuous sequence of saturated models dense with amalgamation bases. Fix  $N \prec_\kappa M$  an amalgamation base of cardinality  $\kappa$ . Let  $p := \text{ga-tp}(a/N)$ . Suppose for the sake of contradiction that  $p$  is not realized in  $M$ .

We can use the assumption that each  $M_i$  is dense with amalgamation bases and the Downward Löwenheim-Skolem axiom to find  $\langle N_i \in \mathcal{K}_\kappa \mid i < \theta \rangle$  an increasing and continuous sequence of amalgamation bases so that  $N \cap M_i \subseteq N_i \prec_\kappa M_i$  for each  $i < \theta$ . Because each  $M_{i+1}$  is  $\kappa^+$ -saturated and dense with amalgamation bases, by Lemma 1 we may further select this

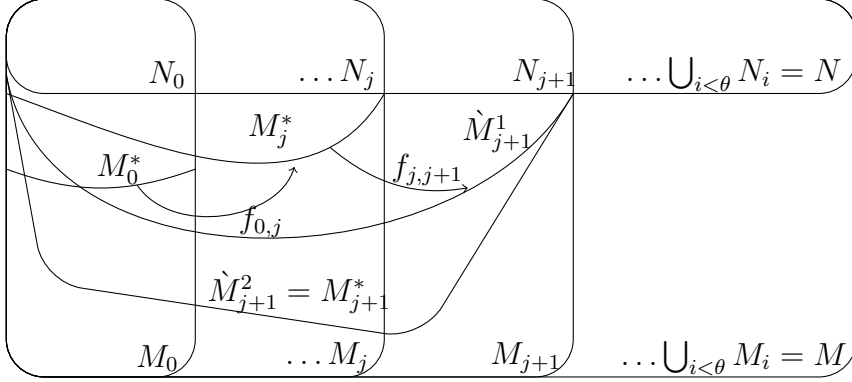


Figure 6: The directed system in the proof of Theorem 1.

sequence so that  $N_{i+1}$  is universal over  $N_i$ . Notice that  $\bigcup_{i<\theta} N_i$  is a  $(\kappa, \theta)$ -limit model and hence an amalgamation base. Because we are assuming that  $a \notin M$ , we know that  $a \notin \bigcup_{i<\theta} N_i$ . This allows us to assume without loss of generality that  $N$  is the  $(\kappa, \theta)$ -limit model  $\bigcup_{i<\theta} N_i$  and  $p := \text{ga-tp}(a/N)$  is a Galois-type omitted in  $M$ .

Then by  $\kappa$ -superstability, we may assume without loss of generality that  $p$  does not  $\kappa$ -split over  $N_0$ , by possibly renumbering the sequences  $\bar{N}$  and  $\bar{M}$ .

For each  $i < \theta$ , because  $M_i$  is  $\kappa^+$ -saturated and dense with amalgamation bases, by Corollary 2 and Proposition 1,  $M_i$  is isomorphic to both a  $(\kappa, \kappa^+)$ -limit model and a  $(\kappa^+, \kappa^+)$ -limit model. So, inside each  $M_i$  we can find a  $(\kappa^+, \kappa^+)$ -limit model witnessed by a sequence that we will denote by  $\langle \dot{M}_i^\alpha \in \mathcal{K}_{\kappa^+} \mid \alpha < \kappa^+ \rangle$ , and we may arrange the enumeration so that  $N_i \prec_{\mathcal{K}} \dot{M}_i^0$ .

We will build a directed system of models  $\langle M_i^* \mid i < \theta \rangle$  with mappings  $\langle f_{i,j} \mid i \leq j < \theta \rangle$  so that the following conditions are satisfied:

1.  $M_i^* \in \mathcal{K}_{\kappa^+}$ .
2.  $M_i^* \preceq_{\mathcal{K}} \bigcup_{\alpha < \kappa^+} \dot{M}_i^\alpha \preceq_{\mathcal{K}} M_i$ .
3. for  $i \leq j < \theta$ ,  $f_{i,j} : M_i^* \rightarrow M_j^*$ .
4. for  $i \leq j < \theta$ ,  $f_{i,j} \upharpoonright N_i = \text{id}_{N_i}$ .
5.  $M_{i+1}^*$  is universal over  $f_{i,i+1}(M_i^*)$ .

Refer to Figure 6.

The construction is possible. Take  $M_0^*$  to be  $\dot{M}_0^1$  and  $f_{0,0} = \text{id}$ . At limit stages take  $M_i^{**}$  and  $\langle f_{k,i}^{**} \mid k < i \rangle$  to be a direct limit as in Fact 5 which is possible because each  $N_i$  is an amalgamation base. We do not immediately get that  $M_i^{**} \preceq_{\mathcal{K}} M_i$ ; we just know we can choose  $M_i^{**}$  to contain  $N_i$  by the continuity of  $\bar{N}$  and condition 4 of the construction. We also know by condition 5 that  $M_i^{**}$  is a  $(\kappa^+, i)$ -limit model witnessed by  $\langle f_{k,i}(M_k^*) \mid k < i \rangle$ . By the uniqueness of limit models of cardinality  $\kappa^+$  and Proposition 1,  $M_i^{**}$  is a  $(\kappa^+, \kappa^+)$ -limit model. Since  $N_i$  has cardinality  $\kappa$ , being able to write  $M_i^{**}$  as a  $(\kappa^+, \kappa^+)$ -limit model tells us that  $M_i^{**}$  is  $\kappa^+$ -universal over  $N_i$ . Recall that  $\bigcup_{\alpha < \kappa^+} \dot{M}_i^\alpha$  is also a  $(\kappa^+, \kappa^+)$ -limit model containing  $N_i$ . Therefore, by a back-and-forth argument, we can find an isomorphism  $g$  from  $M_i^{**}$  to  $\bigcup_{\alpha < \kappa^+} \dot{M}_i^\alpha$  fixing  $N_i$ . Now take  $M_i^* := g(M_i^{**}) = \bigcup_{\alpha < \kappa^+} \dot{M}_i^\alpha$ ,  $f_{k,i} := g \circ f_{k,i}^{**}$  for  $k < i$ , and  $f_{i,i} = \text{id}$ .

For the successor stage of the construction, assume that  $M_j^*$  and  $\langle f_{k,j} \mid k \leq j \rangle$  have been defined. Since  $M_j^*$  is a model of cardinality  $\kappa^+$  containing  $N_j$  and because  $\dot{M}_{j+1}^1$  is  $\kappa^+$ -universal over  $N_{j+1}$  we can find an embedding  $g : M_j^* \rightarrow \dot{M}_{j+1}^1$  with  $g \upharpoonright N_j = \text{id}_{N_j}$ . Take  $M_{j+1}^* := \dot{M}_{j+1}^2$ , set  $f_{k,j+1} := g \circ f_{k,j}$  for all  $k \leq j$ , and define  $f_{j+1,j+1} := \text{id}$ . This completes the construction.

Take  $M^*$  in  $\mathfrak{C}$  with mappings  $\langle f_{i,\theta} \mid i < \theta \rangle$  to be the direct limit of the system as in Fact 5. While  $M^*$  may not be inside  $M$ , we can arrange that  $f_{i,\theta} \upharpoonright N_i = \text{id}_{N_i}$  and that  $N \prec_{\mathcal{K}} M^*$ . Notice that by condition 5 of the construction,  $M^*$  is a  $(\kappa^+, \theta)$ -limit model. By the uniqueness of  $\kappa^+$ -limit models, we know that  $M^*$  is saturated.

For each  $i < \theta$ , let  $f_{i,\theta}^* \in \text{Aut}(\mathfrak{C})$  extend  $f_{i,\theta}$  so that  $f_{i,\theta}^*(N) \preceq_{\mathcal{K}} M^*$ . This is possible since we know that  $M^*$  is  $\kappa^+$ -universal over  $f_{i,\theta}(M_i)$  by condition 5 of the construction. Let  $N^* \prec_{\mathcal{K}} M^*$  be a model of cardinality  $\kappa$  extending  $N$  and  $\bigcup_{i < \theta} f_{i,\theta}^*(N)$ . By the extension property for non- $\kappa$ -splitting, we can find  $p^* \in \text{gaS}(N^*)$  extending  $p$  so that

$$p^* \text{ does not } \kappa\text{-split over } N_0. \quad (7)$$

Since  $M^*$  is a saturated model of cardinality  $\kappa^+$ , we can find  $b^* \in M^*$  realizing  $p^*$ . By the definition of a direct limit, there exists  $0 < i < \theta$  and  $b \in M_i^*$  so that  $f_{i,\theta}(b) = b^*$ .

Because  $f_{i,\theta} \upharpoonright N_i = \text{id}_{N_i}$ , we know that  $b \models p \upharpoonright N_i$ . Suppose for sake of contradiction that there is some  $j > i$  so that  $\text{ga-tp}(b/N_j) \neq p \upharpoonright N_j$ . Then, by the uniqueness of non-splitting extensions, it must be the case that

$\text{ga-tp}(b/N_j)$   $\kappa$ -splits over  $N_0$ . By invariance,

$$\text{ga-tp}(f_{i,\theta}(b)/f_{i,\theta}^*(N_j)) \kappa\text{-splits over } N_0. \quad (8)$$

By monotonicity of non-splitting, the definition of  $b$ , and choice of  $N^*$  containing  $f_{i,\theta}^*(N)$ , (8) implies  $\text{ga-tp}(b^*/N^*)$   $\kappa$ -splits over  $N_0$ . This contradicts (7).

Since  $b \models p \upharpoonright N_j$  for all  $j < \theta$  and  $p \upharpoonright N_j$  does not  $\kappa$ -split over  $N_0$ ,  $\kappa$ -superstability implies that  $\text{ga-tp}(b/N)$  does not  $\kappa$ -split over  $N_0$ . By uniqueness of non- $\kappa$ -splitting extensions  $\text{ga-tp}(b/N) = p$ . Since  $b \in M_i$ , we are done.  $\square$

*Proof of ④  $\Rightarrow$  ① of Theorem 1.* First notice that by Lemma 2 every limit model is dense with amalgamation bases. Next we show that by ④ every limit model of cardinality  $\mu = \kappa^+$  is saturated. To see this consider  $N$  a limit model of cardinality  $\kappa^+$  witnessed by  $\langle N_i \mid i < \theta \rangle$ . By  $\kappa^+$ -applications of Fact 2, for each  $N_i$  we can find  $N'_i$  a  $(\kappa^+, \kappa^+)$ -limit model extending  $N_i$ . By Fact 2 and Proposition 1 each  $N'_i$  is a  $(\kappa, \kappa^+)$ -limit model. Thus each  $N'_i$  is saturated and dense with  $\kappa$ -amalgamation bases. Because  $N_{i+1}$  is universal over  $N_i$  there is  $f_i : N'_i \rightarrow_{N_i} N_{i+1}$ . Let  $N_i^* := f_i(N'_i)$ . Notice that  $\langle N_i^* \mid i < \theta \rangle$  is an increasing sequence of saturated models dense with amalgamation bases and  $N = \bigcup_{i < \theta} N_i^*$ . Thus by our assumption ④,  $N$  is saturated.

To prove ①, suppose that  $\langle M_i \mid i < \theta \rangle$  is an increasing and continuous chain of limit models each of cardinality  $\kappa^+$ . By the previous paragraph we can apply ④ to the sequence  $\langle M_i \mid i < \theta \rangle$  to conclude that  $M := \bigcup_{i < \theta} M_i$  is saturated. By Lemma 3,  $M$  is dense with amalgamation bases. By Corollary 3,  $M$  is an amalgamation base as required.  $\square$

The question remains: Are the assumptions of Theorem 1 enough on their own to prove that the union of an increasing and continuous chain of limit models is an amalgamation base?

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